Density theorems for Hausdorff & packing measures

Definition Let $0\le s<\infty$, $A\subset\mathbb{R}^d$, $a\in A$. The upper and lower s-densities of A at a:

$$\overline{d}^s(A,a):=\overline{\lim_{r\downarrow0}}\frac{1}{(2r)^s}\cdot\mathcal{H}^s(A\cap B(a,r)),$$

$$\underline{d}_s(A,a):=\underline{\lim_{r\to0}}\frac{1}{(2r)^s}\mathcal{H}^s(A\cap B(a,r)).$$

When they agree their common value is the s-dimensional density of A at a and denoted by $d^s(A,a)$.

The following theorem is related to the Lebesgue density thm.

Theorem Let $A\subset\mathbb{R}^d$, $\mathcal{H}^s(A)<\infty$.

(1) $2^{-s}\le\overline{d}^s(A,x)\le1$ for $\mathcal{H}^s$ a.a. $x\in A$.

(2) If A is $\mathcal{H}^s$ measurable then $\overline{d}^s(A,x)=0$ for $\mathcal{H}^s$ a.a. $x\in\mathbb{R}^d\setminus A$.

Corollary Let $A,B\subset\mathbb{R}^d$ be $\mathcal{H}^s$ measurable with $B\subset A$ & $\mathcal{H}^s(A)<\infty$. Then for $\mathcal{H}^s$-a.a. $x\in B$:
$\overline{d}^s(B,x)=\overline{d}^s(A,x)$ & $\underline{d}^s(B,x)=\underline{d}^s(A,x)$.

Proof Apply part (2) to $A\setminus B$.

Remark $\exists C\subset\mathbb{R}^d$ cpt. with $0<\mathcal{H}^s<\infty$ & $\underline{d}^s(C,x)=0$ $\forall x\in\mathbb{R}^d$.

Proof of the Theorem

The proof of the left hand side inequality of (1):
$B=\{x\in A:\overline{d}^s(A,x)<2^{-s}\}$. Then $B=\bigcup_k B_k$, where

$$B_k=\left\{x\in A:\mathcal{H}^s(A\cap B(x,r))<\frac{k}{k+1}r^s \text{ for } 0<r<\frac1k\right\},\ k=1,2,\ldots$$

So, it is enough to show that $\mathcal{H}^s(B_k)=0$, $\forall k$. To see this:
Fix k, let $t:=\frac{k}{k+1}$ and let $\varepsilon>0$. We can find a $\frac1k$-cover $\{E_i\}_{i=1}^\infty$ of B_k s.t.

$B_k\subset\bigcup_i E_i$, $|E_i|<\frac1k$, $B_k\cap E_i\neq\emptyset$,
$\sum_i|E_i|^s<\mathcal{H}^s(B_k)+\varepsilon$.

For every i: let $x_i\in B_k\cap E_i$, $r_i:=|E_i|$.

Then $B_k\cap E_i\subset A\cap B(x_i,r_i)$ and

$$\mathcal{H}^s(B_k)=\sum_i\mathcal{H}^s(B_k\cap E_i)\le\sum_i\mathcal{H}^s(A\cap B(x_i,r_i))<\sum_i t\,r_i^s=t\sum_i|E_i|^s\le t(\mathcal{H}^s(B_k)+\varepsilon).$$

Let $\varepsilon\downarrow0$. We get $\mathcal{H}^s(B_k)\le t\mathcal{H}^s(B_k)$. But $\mathcal{H}^s(B_k)<\infty$ & $t<1$. So this can happen only if $\mathcal{H}^s(B_k)=0$.

Now we prove the right hand side inequality in part (1):
First observe that we may assume that A is a Borel set for the Borel regularity of $\mathcal{H}^s$. Let $t>1$,
$B:=\{x\in A:\overline{d}^s(A,x)>t\}$.
It is enough to show that $\mathcal{H}^s(B)=0$.

Let $\varepsilon,\delta>0$. We know that $\mathcal{H}^s|_A$ is a Radon measure. Hence we can choose an open set U s.t. $B\subset U$ & $\mathcal{H}^s(A\cap U)<\mathcal{H}^s(B)+\varepsilon$.
For $\forall x\in B$ $\exists$ arbitrarily small $0<r<\delta/2$, s.t. $B(x,r)\subset U$ & $\mathcal{H}^s(A\cap B(x,r))>t(2r)^s$.
Alkalmazva a "Vitali lefedési tétel Radon mértékekre" tételt: $\exists\{B_i\}_{i=1}$ disjoint balls: $\mathcal{H}^s(B\setminus\bigcup_i B_i)=0$.

$$\mathcal{H}^s(B)+\varepsilon>\mathcal{H}^s(A\cap U)\ge\sum_i\mathcal{H}^s(A\cap B_i)>t\sum_i|B_i|^s\ge t\mathcal{H}^s_\delta\Big(B\cap\bigcup_i B_i\Big)=\mathcal{H}^s_\delta(B)\cdot t.$$

This holds because $\mathcal{H}^s_\delta(B\setminus\bigcup_i B_i)=0$ and
$\mathcal{H}^s_\delta(B)\le\underbrace{\mathcal{H}^s_\delta(B\setminus\bigcup_i B_i)}_{0}+\underbrace{\mathcal{H}^s_\delta(B\cap\bigcup_i B_i)}_{\le\mathcal{H}^s_\delta(B)}$
here we used the subadditivity of $\mathcal{H}^s_\delta$. Let $\varepsilon\downarrow0$, $\delta\downarrow0$ and use that $t>1$ to get that $\mathcal{H}^s(B)=0$.

Now we prove part (2): Let $t>0$.
$B:=\{x\in\mathbb{R}^d\setminus A:\overline{d}^s(A,x)>t\}$.
We have to prove that $\mathcal{H}^s(B)=0$.
Let $\varepsilon>0$. We know that $\mathcal{H}^s|_A(B)=0$.
So $\exists$ U open set s.t. $B\subset U$ & $\mathcal{H}^s(A\cap B)<\varepsilon$.
$\forall x\in B$ $\exists r(x)>0$ s.t. $B(x,r(x))\subset U$,
$\mathcal{H}^s(A\cap B(x,r(x)))>t(2\cdot r(x))^s$.
Using the "5r covering thm" $\exists x_1,x_2,\ldots\in B$ s.t.

- $B_i:=B(x_i,r(x_i))$ are disjoint
- $\{5B_i\}$ cover B

Then

$$t\mathcal{H}^s_\infty(B)\le t\cdot\sum_i|5B_i|^s=t\cdot5^s\sum_i|B_i|^s<5^s\cdot\sum_i\mathcal{H}^s(A\cap B_i)\le5^s\mathcal{H}^s(A\cap U)<5^s\cdot\varepsilon.$$

Let $\varepsilon\downarrow0$ to get $\mathcal{H}^s_\infty(B)=0$.
This implies that $\mathcal{H}^s(B)=0$.

Theorem Let μ be a Radon measure on $\mathbb{R}^d$ s.t. $0<\mu(\mathbb{R}^d)<\infty$. Let $F\subset\mathbb{R}^d$ be a Borel set and $0<c<\infty$ be a constant.

(a) If $\forall x\in F$, $\underline{\lim}_{r\downarrow0}\frac{\mu(B(x,r))}{r^s}<c$ then $\mathcal{H}^s(F)\ge\frac{\mu(F)}{c}$.

(b) If $\forall x\in F$, $\overline{\lim}_{r\downarrow0}\frac{\mu(B(x,r))}{r^s}>c$ then $\mathcal{H}^s(F)\le2^s\cdot\frac{\mu(\mathbb{R}^d)}{c}$.

Proof (a) $\forall\delta>0$ let
$F_\delta=\{x\in F:\frac{\mu(B(x,r))}{r^s}<c,\ \forall 0<r\le\delta\}$.

Let $\{U_i\}$ be a δ-cover of F and so of F_δ. If $U_i\cap F_\delta\neq\emptyset$ then we pick $x\in U_i\cap F_\delta$. Let $B:=B(x,|U_i|)$. Clearly, $U_i\subset B$.
$\mu(U_i)\le\mu(B)<c|U_i|^s$ (by the def of F_δ).
Hence
$\mu(F_\delta)\le\sum_i\{\mu(U_i):U_i\cap F_\delta\neq\emptyset\}\le c\cdot\sum_i|U_i|^s$.
Using that $\{U_i\}$ is any δ-cover of F
$\mu(F_\delta)\le c\mathcal{H}^s_\delta(F)\le c\mathcal{H}^s(F)$. Since $F_\delta\nearrow F$ as $\delta\downarrow0$ we get $\mu(F)\le c\mathcal{H}^s(F)$. □

Proof of (b) We prove this part only for 2^s replaced with 8^s.
We may assume that F is bounded. Fix $\delta>0$ and let
$\mathcal{C}:=\{B(x,r):x\in F,0<r\le\delta\ \&\ \frac{\mu(B(x,r))}{r^s}>c\}$.
By assumption $F\subset\bigcup_{B\in\mathcal{C}}B$.
Using the 5r-covering thm:
$\exists\{B_i\}_i$ disjoint balls s.t.
· $B_i\in\mathcal{C}$ & $\bigcup_i5B_i\supset$

So $\{5B_i\}$ is a 10δ-cover of F. Hence,

$$\mathcal{H}^s_{10\delta}(F)\le\sum_i|5B_i|^s\le5^s\sum_i|B_i|^s\le10^sc^{-1}\sum_i\mu(B_i)\le\le10^sc^{-1}\mu(\mathbb{R}^d).$$

Let $\delta\downarrow0$ to get $\mathcal{H}^s(F)\le10^sc^{-1}\mu(\mathbb{R}^d)<\infty$.

Definition: local dimension of a measure

Let μ be a Radon measure on $\mathbb{R}^d$. Assume that $x\in spt(\mu)$ that is $\mu(B(x,r))>0$ $\forall r>0$.
then we define the lower and upper local dimension of μ at x by

$$\underline{\dim}_{loc}(\mu,x):=\underline{\lim_{r\downarrow0}}\frac{\log\mu(B(x,r))}{\log r},$$

$$\overline{\dim}_{loc}(\mu,x):=\overline{\lim_{r\downarrow0}}\frac{\log\mu(B(x,r))}{\log r}.$$

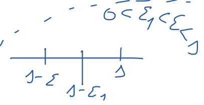
If $x\notin spt(\mu)$ then we define $\underline{\dim}_{loc}(\mu,x)=\overline{\dim}_{loc}(\mu,x)=\infty$.

Theorem Let $E\subset\mathbb{R}^d$ be Borel and let μ be a finite Radon measure. Then

(a) If $\mu(E)>0$ & $\forall x\in E$ we have $\underline{\dim}_{loc}(\mu,x)\ge s$ then $\dim_H E\ge s$.

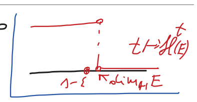
(b) If $\forall x\in E$ $\overline{\dim}_{loc}(\mu,x)\le s$ then $\dim_H E\le s$.

Proof: part (a)



If $\underline{\dim}_{loc}(\mu,x)>s-\varepsilon_1$, then $\exists r_0$ s.t. $\forall 0<r<r_0$:
$\frac{\log\mu(B(x,r))}{\log r}>s-\varepsilon_1$, hence
$\mu(B(x,r))<r^{s-\varepsilon_1}=r^{s-\varepsilon}\cdot r^{\varepsilon-\varepsilon_1}$
So $\frac{\mu(B(x,r))}{r^{s-\varepsilon}}<r^{\varepsilon-\varepsilon_1}\searrow$ as $r\downarrow0$.
Hence $\underline{\lim}_{r\downarrow0}\frac{\mu(B(x,r))}{r^{s-\varepsilon}}<c$ $\forall c>0$.
By the prev. thm: $\mathcal{H}^{s-\varepsilon}(E)\ge\frac{\mu(E)}{c}$ holds for $\forall c>0$.

So $\mathcal{H}^{s-\varepsilon}(E)=\infty$
$s-\varepsilon\le\dim_H E$
holds for $\forall\varepsilon>0$ □



Now we prove part (b)
Let $\varepsilon>0$. Then $\exists r_n\downarrow0$
$\frac{\log\mu(B(x,r_n))}{\log r_n}<s+\varepsilon$. That is
$\mu(B(x,r_n))>r_n^{s+\varepsilon}$. So,
$\frac{\mu(B(x,r_n))}{r_n^{s+\varepsilon}}>1$. Hence
$\overline{\lim}_{r\downarrow0}\frac{\mu(B(x,r))}{r^{s+\varepsilon}}>1$ holds for $\forall x\in E$.
Then by the previous thm: (holds $\forall\varepsilon>0$)
$\mathcal{H}^{s+\varepsilon}(E)\le2^{s+\varepsilon}\frac{\mu(E)}{1}\Rightarrow\dim_H E\le s+\varepsilon$ □